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On certain Schlömilch-type series

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Abstract

By using a form of the Poisson summation formula together with a generalization due to Srivastava and Exton of the discontinuous integral of Weber and Schafheitlin and the (heretofore not readily available) cosine transform of the hypergeometric function ${}_1F_2[-b^2t^2]$ $(b>0)$, several new Schlömilch and Fourier-type series are evaluated. By specialization of the latter series numerous results appearing in the literature are obtained in a unified way.

Keywords: Schlömilch and Fourier-type series; Bessel functions; Series of hypergeometric functions

AMS classifications: 33C10; 33C20; 42A24

1. Introduction

Recently, Rawn [9] by using Fourier and Hankel transform techniques and a complex contour integration method deduced a number of interesting summation formulas for one-dimensional Schlömilch and Fourier-type series. For example,

$$\sum_{n=1}^{\infty}(-1)^n\frac{J_\mu(2nx)}{n^\mu}=-\frac{x^\mu}{2\Gamma(\mu+1)}, \tag{1.1}$$

where $\operatorname{Re}\mu>-\frac{1}{2}$, $-\frac{1}{2}\pi<x<\frac{1}{2}\pi$. This result is a special case of Nielsen's summation formula [13, p. 636, Eq. (4)]

$$\sum_{n=1}^{\infty}(-1)^n\frac{J_\mu(2nx)}{n^\mu}=-\frac{x^\mu}{2\Gamma(\mu+1)}+\frac{\sqrt{\pi}\,x^{-\mu}}{\Gamma(\mu+\frac{1}{2})}\sum_{k=1}^{p}\left[x^2-\left(k-\frac{1}{2}\right)^2\pi^2\right]^{\mu-1/2}, \tag{1.2}$$

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where $\operatorname{Re}\mu > -\frac{1}{2}$, $x > 0$ and p is a nonnegative integer such that $(p-\frac{1}{2})\pi < x < (p+\frac{1}{2})\pi$. When $0 < x < \frac{1}{2}\pi$, then $p = 0$ and the second summation vanishes by definition. Thus, if we divide the resulting equation by x^μ, both sides become even functions of x (see Eq. (1.3) below), and further if we agree that the left-hand side of this new equation is Cesàro-summable at $x=0$ we obtain Eq. (1.1). Schlömilch-type series in one and more dimensions are not only of interest in pure mathematics [4], but have utility in applied mathematics and mathematical physics (see, e.g., [1, 2, 5]).

By letting $z = nx$ in

$$\frac{J_\mu(2z)}{z^\mu} = \frac{{}_0F_1[-;\mu+1;-z^2]}{\Gamma(\mu+1)}, \tag{1.3}$$

we obtain from Eq. (1.2) Nielsen's formula in hypergeometric form:

$$\sum_{n=1}^{\infty}(-1)^n {}_0F_1[-;\mu+1;-n^2x^2] = -\frac{1}{2} + \frac{\sqrt{\pi}}{x^{2\mu}}\frac{\Gamma(\mu+1)}{\Gamma(\mu+\frac{1}{2})}\sum_{k=1}^{p}\left[x^2 - \left(k-\frac{1}{2}\right)^2\pi^2\right]^{\mu-1/2}, \tag{1.4}$$

where $\operatorname{Re}\mu > -\frac{1}{2}$, $x > 0$ and p is a nonnegative integer such that $(p-\frac{1}{2})\pi < x < (p+\frac{1}{2})\pi$. In the present investigation, we shall generalize the above summations by essentially replacing the Bessel function in Eq. (1.1) by a finite product of Bessel functions, and in Eq. (1.4) by replacing ${}_0F_1[-n^2x^2]$ by ${}_1F_2[-n^2x^2]$ which contains the latter hypergeometric function as a special case. Thus, we shall not only be able to obtain many of the results in [9], but also deduce several new summation formulas heretofore not available.

In what follows, we shall rely on a form of the Poisson summation formula [12, p. 60, Eq. (2.8.1)], namely,

$$\sum_{n=0}^{\infty} f(n) = \frac{1}{2}f(0) + \int_0^\infty f(t)\,\mathrm{d}t + 2\sum_{k=1}^{\infty}\int_0^\infty \cos(2\pi kt)f(t)\,\mathrm{d}t, \tag{1.5}$$

where the penultimate integral exists and $f(t)$ is continuous and of bounded variation in $(0,\infty)$. As we shall see, this result is especially useful when the second integral on the right is discontinuous and vanishes for all $k > p$ for some integer p. For conciseness, we define for positive numbers y, x_i $(i = 1,\ldots,m)$

$$S_1(y;x_1,\ldots,x_m) \equiv \sum_{n=1}^{\infty}(-1)^n\frac{J_\mu(2ny)}{(ny)^\mu}\prod_{i=1}^{m}\frac{J_{\nu_i}(2nx_i)}{(nx_i)^{\nu_i}}, \tag{1.6}$$

$$S_2(y;x_1,\ldots,x_m) \equiv \sum_{n=1}^{\infty}\frac{J_\mu(2ny)}{(ny)^\mu}\prod_{i=1}^{m}\frac{J_{\nu_i}(2nx_i)}{(nx_i)^{\nu_i}}, \tag{1.7}$$

where for absolute convergence of both series $\operatorname{Re}(\mu+\nu_1+\cdots+\nu_m) > \frac{1}{2} - \frac{1}{2}m$ which may be easily shown by replacing the Bessel functions by appropriate asymptotic forms in the "tail end" of each series. When $m = 0$, we assume the products in Eqs. (1.6) and (1.7) are empty and equal unity.

2. Evaluations of $S_1(y;x_1,\dots,x_m)$

Since for n an integer $(-1)^n=\cos(\pi n)$, by using Eq. (1.5) and noting Eq. (1.3) we have from Eq. (1.6),

$$S_1(y;x_1,\dots,x_m)=-\frac{1}{2\Gamma(\mu+1)}\prod_{i=1}^{m}\frac{1}{\Gamma(\nu_i+1)}+\int_0^\infty\cos(\pi t)\frac{J_\mu(2yt)}{(yt)^\mu}\prod_{i=1}^{m}\frac{J_{\nu_i}(2x_it)}{(x_it)^{\nu_i}}\,\mathrm{d}t$$

$$+2\sum_{k=1}^{\infty}\int_0^\infty\cos(\pi t)\cos(2\pi kt)\frac{J_\mu(2yt)}{(yt)^\mu}\prod_{i=1}^{m}\frac{J_{\nu_i}(2x_it)}{(x_it)^{\nu_i}}\,\mathrm{d}t.$$

A brief computation involving a trigonometric identity and adjustment of summation indicies then gives

$$S_1(y;x_1,\dots,x_m)=-\frac{1}{2\Gamma(\mu+1)}\prod_{i=1}^{m}\frac{1}{\Gamma(\nu_i+1)}$$

$$+2\sum_{k=1}^{\infty}\int_0^\infty\cos(\pi(2k-1)t)\frac{J_\mu(2yt)}{(yt)^\mu}\prod_{i=1}^{m}\frac{J_{\nu_i}(2x_it)}{(x_it)^{\nu_i}}\,\mathrm{d}t. \tag{2.1}$$

Now, noting that $\cos z=\sqrt{\pi z/2}\;J_{-1/2}(z)$, we obtain

$$S_1(y;x_1,\dots,x_m)=-\frac{1}{2\Gamma(\mu+1)}\prod_{i=1}^{m}\frac{1}{\Gamma(\nu_i+1)}+\pi\sqrt{2}\;y^{-\mu}\prod_{i=1}^{m}x_i^{-\nu_i}$$

$$\times\sum_{k=1}^{\infty}\sqrt{2k-1}\int_0^\infty t^{(1/2)-\mu-\nu_1-\cdots-\nu_m}\prod_{i=1}^{m}J_{\nu_i}(2x_it)J_\mu(2yt)J_{-1/2}(\pi(2k-1)t)\,\mathrm{d}t. \tag{2.2}$$

The latter infinite integral is a special case of a generalization of the discontinuous integral of Weber and Schafheitlin (see [13, p. 398 et seq.]) which has been evaluated by Srivastava and Exton [10, Eq. (2.8)] who also show that the integral vanishes [10, Eq. (3.4)] for real $k>1/2$ and nonnegative integers m provided that

$$\pi(k-\tfrac{1}{2})>y+x_1+\cdots+x_m, \tag{2.3a}$$

$$\mathrm{Re}(\mu+\nu_1+\cdots+\nu_m)>-\tfrac{1}{2}-\tfrac{1}{2}m, \tag{2.3b}$$

which secure the convergence of the integral and the conditional convergence of $S_1(y;x_1,\dots,x_m)$. In [10] it was mentioned that Bailey also showed that the integral in Eq. (2.2) vanishes under the above conditional inequalities.

Since the inequality (2.3a) will hold for all integers $k>1$ if, in particular, it holds for $k=1$, we find from Eqs. (1.6) and (2.2) the result

$$\sum_{n=1}^{\infty}(-1)^n\frac{J_\mu(2ny)}{n^\mu}\prod_{i=1}^{m}\frac{J_{\nu_i}(2nx_i)}{n^{\nu_i}}=-\frac{y^\mu}{2\Gamma(\mu+1)}\prod_{i=1}^{m}\frac{x_i^{\nu_i}}{\Gamma(\nu_i+1)}, \tag{2.4}$$

where $y + x_1 + \cdots + x_m < \frac{1}{2}\pi$ and $\mathrm{Re}(\mu + v_1 + \cdots + v_m) < -\frac{1}{2} - \frac{1}{2}m$. When $m = 0$, we deduce immediately Eq. (1.1), and also obtain by specialization:

$$\sum_{n=1}^{\infty}(-1)^n\frac{J_\mu(2ny)\sin^m(nx)}{n^{\mu+m}} = -\frac{x^m y^\mu}{2\Gamma(\mu+1)},\quad \mathrm{Re}\,\mu > -\tfrac{1}{2} - m,\ \tfrac{1}{2}m|x| + |y| < \tfrac{1}{2}\pi, \tag{2.5}$$

$$\sum_{n=1}^{\infty}(-1)^n\frac{\sin^m(ny)}{n^m} = -\frac{1}{2}y^m,\quad |y| < \pi/m, \tag{2.6}$$

$$\sum_{n=1}^{\infty}(-1)^n\frac{J_0(ny)\sin^m(ny)}{n^m} = -\frac{1}{2}y^m,\quad |y| < \pi/(m+1), \tag{2.7}$$

$$\sum_{n=1}^{\infty}(-1)^n\frac{J_\mu(2ny)J_v(2nx)}{n^{\mu+v}} = -\frac{y^\mu x^v}{2\Gamma(\mu+1)\Gamma(v+1)},\quad \mathrm{Re}(\mu+v) > -1,\quad |x| + |y| < \tfrac{1}{2}\pi, \tag{2.8}$$

$$\sum_{n=1}^{\infty}(-1)^n\frac{J_\mu(2ny)J_v(2ny)}{n^{\mu+v}} = -\frac{y^{\mu+v}}{2\Gamma(\mu+1)\Gamma(v+1)},\quad \mathrm{Re}(\mu+v) > -1,\quad |y| < \tfrac{1}{4}\pi. \tag{2.9}$$

In the latter specializations, by applying the argument just below Eq. (1.2), we have replaced in the conditional inequalities, for example, y by $|y|$ and assumed Cesàro summability when $y = 0$. Eqs. (2.5)–(2.9) are derived in [9], where the pertinent attributions are cited in the references. However, Eq. (2.6) was attributed to R. Butler who in fact studied the series without the factor $(-1)^n$. A generalization of Eq. (2.6) is recorded in [8, vol. 1, p. 744, Eq. (19)]. Furthermore, the conditions of validity for Eqs. (2.5)–(2.9) and other results recorded in [9] are not in some cases best possible or always correct. For example, for Eq. (2.5) the conditions given there are $|\frac{m}{2}x + y| < \frac{1}{2}\pi$, $\mathrm{Re}\,\mu > -\frac{3}{2}$.

Glasser [3, Eq. (14)] has obtained Eq. (2.9) as well, but the condition $0 < x < 2$ for its validity contains a misprint which should read $0 < x < \frac{1}{2}\pi$. Without too much of additional work we shall obtain a more general result which contains Eq. (2.9) as a special case. Thus, from Eqs. (2.1) and (2.2) it is easy to see that

$$S_1(y,y) = -\frac{1}{2\Gamma(\mu+1)\Gamma(v+1)} + \frac{2}{y^{\mu+v}}\sum_{k=1}^{\infty}\int_0^\infty \frac{\cos(\pi(2k-1)t)J_\mu(2yt)J_v(2yt)}{t^{\mu+v}}\,\mathrm{d}t \tag{2.10}$$

$$= -\frac{1}{2\Gamma(\mu+1)\Gamma(v+1)} + \frac{\pi\sqrt{2}}{y^{\mu+v}}\sum_{k=1}^{\infty}\sqrt{2k-1}\int_0^\infty t^{1/2-\mu-v}J_{-1/2}(\pi(2k-1)t)J_\mu(2yt)J_v(2yt)\,\mathrm{d}t,$$

where $\mathrm{Re}(\mu+v) > -1$. But by the inequalities (2.3) either integral vanishes for all positive integers k such that $y < \frac{1}{2}\pi(k - \frac{1}{2})$. Thus, we may write

$$S_1(y,y) = -\frac{1}{2\Gamma(\mu+1)\Gamma(v+1)} + \frac{2}{y^{\mu+v}}\sum_{k=1}^{p}\int_0^\infty \frac{\cos(\pi(2k-1)t)J_\mu(2yt)J_v(2yt)}{t^{\mu+v}}\,\mathrm{d}t,$$

where p is a nonnegative integer such that $(p - \frac{1}{2})\frac{1}{2}\pi < y < (p + \frac{1}{2})\frac{1}{2}\pi$.

The integral in the latter equation may be evaluated by using [8, vol. 2, p. 226, Eq. (11)]. Thus, after some algebra involving also elementary gamma function identities, we find for $\operatorname{Re}(\mu+\nu) > -1$, $y > 0$:

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^n\frac{J_\mu(2ny)J_\nu(2ny)}{n^{\mu+\nu}} &= -\frac{y^{\mu+\nu}}{2\Gamma(\mu+1)\Gamma(\nu+1)}\\ &+\frac{\sqrt{\pi}\,\Gamma(\mu+\nu)y^{\mu+\nu-1}}{\Gamma(\mu+\frac{1}{2})\Gamma(\nu+\frac{1}{2})\Gamma(\mu+\nu+\frac{1}{2})}\\ &\times\sum_{k=1}^{p}{}_3F_2\left[\begin{matrix}\frac{1}{2}-\mu-\nu,\frac{1}{2}-\mu,\frac{1}{2}-\nu; \\ \frac{1-\mu-\nu}{2},\frac{2-\mu-\nu}{2};\end{matrix}\ \frac{\pi^2(k-\frac{1}{2})^2}{4y^2}\right]\\ &+\frac{\pi^{\mu+\nu+1/2}}{2y}\,\frac{\Gamma(-\frac{\mu+\nu}{2})}{\Gamma(\frac{1-\mu+\nu}{2})\Gamma(\frac{1+\mu-\nu}{2})\Gamma(\frac{1+\mu+\nu}{2})}\\ &\times\sum_{k=1}^{p}\left(k-\frac{1}{2}\right)^{\mu+\nu}{}_4F_3\left[\begin{matrix}\frac{1-\mu-\nu}{2},\frac{1+\mu-\nu}{2},\frac{1-\mu+\nu}{2},\frac{1+\mu+\nu}{2}; \\ \frac{1}{2},\frac{\mu+\nu}{2},\frac{2+\mu+\nu}{2};\end{matrix}\ \frac{\pi^2(k-\frac{1}{2})^2}{4y^2}\right]\\ &+\frac{2^{\mu+\nu}\pi^{\mu+\nu+1}(\mu+\nu)}{y^2}\,\frac{\sin(\frac{1}{2}\pi(\mu+\nu))\Gamma(-1-\mu-\nu)}{\Gamma(\frac{\mu-\nu}{2})\Gamma(\frac{\nu-\mu}{2})}\\ &\times\sum_{k=1}^{p}\left(k-\frac{1}{2}\right)^{\mu+\nu+1}{}_3F_2\left[\begin{matrix}\frac{2-\mu-\nu}{2},\frac{2+\mu-\nu}{2},\frac{2-\mu+\nu}{2}; \\ \frac{3}{2},\frac{3+\mu+\nu}{2};\end{matrix}\ \frac{\pi^2(k-\frac{1}{2})^2}{4y^2}\right],\end{aligned} \tag{2.11}$$

where p is a nonnegative integer such that $(p-\frac{1}{2})\frac{1}{2}\pi < y < (p+\frac{1}{2})\frac{1}{2}\pi$. When $0 < y < \frac{1}{4}\pi$, then $p=0$, there are no contributions from the three k-summations, and the result reduces to Eq. (2.9). Thus, the assertion in [3] that Eq. (2.9) is valid for $0 < y < 1$ cannot be correct. Rawn [9] has already noted this, but did not mention that the discrepancy is due to the misprint in [3] noted earlier.

Upon letting $\mu = \nu = \frac{1}{2}$ and noting that $J_{1/2}(z) = \sqrt{2/\pi z}\sin z$, it is not too difficult to show that Eq. (2.11) reduces to

$$\sum_{n=1}^{\infty}(-1)^n\frac{\sin^2(ny)}{n^2} = -\frac{1}{2}(y-\pi p)^2,$$

where $y > 0$ and p is a nonnegative integer such that $(p-\frac{1}{2})\pi < y < (p+\frac{1}{2})\pi$. The latter equation and Eq. (2.6) specialized with $m = 2$ are of course equivalent since their left members are even functions of y whose period is π.

3. Evaluations of $S_2(y;x_1,\dots,x_m)$

From Eqs. (1.5) and (1.7) we see upon noting Eq. (1.3) that

$$\begin{aligned} S_2(y;x_1,\dots,x_m) = &-\frac{1}{2\Gamma(\mu+1)}\prod_{i=1}^{m}\frac{1}{\Gamma(v_i+1)} \\ &+y^{-\mu}\prod_{i=1}^{m}x_i^{-v_i}\int_0^\infty t^{-\mu-v_1-\cdots-v_m}J_\mu(2yt)\prod_{i=1}^{m}J_{v_i}(2x_it)\,\mathrm{d}t \\ &+2y^{-\mu}\prod_{i=1}^{m}x_i^{-v_i}\sum_{k=1}^{\infty}\int_0^\infty t^{-\mu-v_1-\cdots-v_m}\cos(2\pi kt)J_\mu(2yt)\prod_{i=1}^{m}J_{v_i}(2x_it)\,\mathrm{d}t. \end{aligned} \tag{3.1}$$

By the same argument used to show that the integrals in Eqs. (2.1) and (2.2) vanish, the latter integral vanishes for all integers $k \geqslant 1$ provided that $\mathrm{Re}(\mu+v_1+\cdots+v_m) > -\frac{1}{2}-\frac{1}{2}m$ and $\pi > y+x_1+\cdots+x_m$. Furthermore, if $y > x_1+\cdots+x_m$ we may again apply the theorem of Srivastava and Exton [10, Eq. (2.8)] to evaluate the penultimate integral in Eq. (3.1). Thus, we deduce from Eqs. (1.7) and (3.1) that for $m \geqslant 1$ and positive numbers y, x_i $(i = 1,2,\dots,m)$

$$\begin{aligned} &\sum_{n=1}^{\infty}\frac{J_\mu(2ny)}{n^\mu}\prod_{i=1}^{m}\frac{J_{v_i}(2nx_i)}{n^{v_i}} \\ &=\frac{1}{2}\prod_{i=1}^{m}\frac{x_i^{v_i}}{\Gamma(v_i+1)}\left\{\frac{\sqrt{\pi}y^{\mu-1}}{\Gamma(\mu+\frac{1}{2})}F_C^{(m)}\left[\frac{1}{2},\frac{1}{2}-\mu;1+v_1,\dots,1+v_m;\frac{x_1^2}{y^2},\dots,\frac{x_m^2}{y^2}\right]\right. \\ &\quad\left.-\frac{y^\mu}{\Gamma(\mu+1)}\right\}, \end{aligned} \tag{3.2}$$

where $\mathrm{Re}(\mu+v_1+\cdots+v_m) > -\frac{1}{2}-\frac{1}{2}m$, $x_1+\cdots+x_m < y$, $x_1+\cdots+x_m+y < \pi$, and $F_C^{(m)}$ is one of Lauricella's hypergeometric functions of m variables (see also, e.g., [11, p. 60, Eq. (3)]) whose convergence is secured by the above second inequality. When $m=1$, the Lauricella function $F_C^{(m)}$ in Eq. (3.2) reduces to the Gaussian hypergeometric function ${}_2F_1$ and we obtain Eqs. (3.5a) and (3.5b) below. We shall discuss the case $m=1$, $x_1=y$ in greater detail at the end of this section (see Eq. (3.7) below).

Since $F_C^{(2)} = F_4$, where F_4 is an Appell function, Eq. (3.2) gives, when $m=2$,

$$\begin{aligned} &\sum_{n=1}^{\infty}\frac{J_\mu(2ny)J_v(2nx)J_\omega(2nz)}{n^{\mu+v+\omega}} \\ &=\frac{x^vz^\omega}{2\Gamma(v+1)\Gamma(\omega+1)}\left\{\frac{\sqrt{\pi}y^{\mu-1}}{\Gamma(\mu+\frac{1}{2})}F_4\left[\frac{1}{2},\frac{1}{2}-\mu;1+v,1+\omega;\frac{x^2}{y^2},\frac{z^2}{y^2}\right]-\frac{y^\mu}{\Gamma(\mu+1)}\right\}, \end{aligned}$$

where $\mathrm{Re}(\mu+v+\omega) > -\frac{3}{2}$, $x+z<y$, $x+y+z<\pi$. Furthermore, if we set $z=x$ and use a reduction formula for F_4 due to Burchnall [11, p. 55, Eq. (16)], we obtain the interesting result

$$\sum_{n=1}^{\infty}\frac{J_\mu(2ny)J_v(2nx)J_\omega(2nx)}{n^{\mu+v+\omega}} = -\frac{y^\mu x^{v+\omega}}{2\Gamma(\mu+1)\Gamma(v+1)\Gamma(\omega+1)}$$

$$+\frac{\sqrt{\pi}y^{\mu-1}x^{\nu+\omega}}{2\Gamma(\mu+\frac{1}{2})\Gamma(\nu+1)\Gamma(\omega+1)}$$
$$\times {}_4F_3\left[\begin{matrix}\frac{1}{2}, \frac{1}{2}-\mu, \frac{1+\nu+\omega}{2}, \frac{2+\nu+\omega}{2}; \\ 1+\nu, 1+\omega, 1+\nu+\omega;\end{matrix} \frac{4x^2}{y^2}\right],$$

where $2x < y$, $2x + y < \pi$, $\mathrm{Re}(\mu+\nu+\omega) > -\frac{3}{2}$.

Other specializations of Eq. (3.2) (including the case $m = 1$) are recorded below:

$$\sum_{n=1}^{\infty} \frac{\sin(2ny)}{n} \prod_{i=1}^{m} \frac{J_{\nu_i}(2nx_i)}{n^{\nu_i}} = \left(\frac{\pi}{2} - y\right) \prod_{i=1}^{m} \frac{x_i^{\nu_i}}{\Gamma(\nu_i+1)},$$
$$\mathrm{Re}(1+\nu_1+\cdots+\nu_m) > -\frac{m}{2}, \quad x_1+\cdots+x_m < y, \quad x_1+\cdots+x_m+y<\pi, \tag{3.3a}$$

$$\sum_{n=1}^{\infty} \frac{\sin(ny)\sin^m(nx)}{n^{m+1}} = \frac{1}{2}x^m(\pi - y), \quad mx < y, \quad mx + y < 2\pi, \tag{3.3b}$$

$$\sum_{n=1}^{\infty} \cos(2ny) \prod_{i=1}^{m} \frac{J_{\nu_i}(2nx_i)}{n^{\nu_i}} = -\frac{1}{2} \prod_{i=1}^{m} \frac{x_i^{\nu_i}}{\Gamma(\nu_i+1)},$$
$$\mathrm{Re}(\nu_1+\cdots+\nu_m) > -\frac{m}{2}, \quad x_1+\cdots+x_m < y, \quad x_1+\cdots+x_m+y<\pi, \tag{3.4a}$$

$$\sum_{n=1}^{\infty} \frac{\cos(ny)\sin^m(nx)}{n^{m}} = -\frac{1}{2}x^m, \quad mx < y, \quad mx + y < 2\pi, \tag{3.4b}$$

$$\sum_{n=1}^{\infty} \frac{J_\mu(2ny)J_\nu(2nx)}{n^{\mu+\nu}} = \frac{y^\mu x^\nu}{2\Gamma(\nu+1)}\left\{\frac{\sqrt{\pi}}{y\Gamma(\mu+\frac{1}{2})}\, {}_2F_1\left[\begin{matrix}\frac{1}{2}, \frac{1}{2}-\mu; \\ 1+\nu;\end{matrix} \frac{x^2}{y^2}\right] - \frac{1}{\Gamma(\mu+1)}\right\},$$
$$\mathrm{Re}(\mu+\nu) > -1, \quad x < y, \quad x + y < \pi, \tag{3.5a}$$

$$\sum_{n=1}^{\infty} \frac{J_\mu(2ny)J_\nu(2nx)}{n^{\mu+\nu}} = \frac{y^\mu x^\nu}{2\Gamma(\mu+1)}\left\{\frac{\sqrt{\pi}}{x\Gamma(\nu+\frac{1}{2})}\, {}_2F_1\left[\begin{matrix}\frac{1}{2}, \frac{1}{2}-\nu; \\ 1+\mu;\end{matrix} \frac{y^2}{x^2}\right] - \frac{1}{\Gamma(\nu+1)}\right\},$$
$$\mathrm{Re}(\mu+\nu) > -1, \quad y < x, \quad x + y < \pi, \tag{3.5b}$$

$$\sum_{n=1}^{\infty} \frac{J_\mu(2ny)J_\nu(2ny)}{n^{\mu+\nu}} = \frac{\sqrt{\pi}\Gamma(\mu+\nu)y^{\mu+\nu-1}}{2\Gamma(\mu+\frac{1}{2})\Gamma(\nu+\frac{1}{2})\Gamma(\mu+\nu+\frac{1}{2})} - \frac{y^{\mu+\nu}}{2\Gamma(\mu+1)\Gamma(\nu+1)},$$
$$\mathrm{Re}(\mu+\nu) > 0, \quad y < \tfrac{1}{2}\pi. \tag{3.5c}$$

Eq. (3.3a) is obtained from Eq. (3.2) by setting $\mu=\frac{1}{2}$ in the latter and noting that the Lauricella function reduces to unity. Now setting $v_i=\frac{1}{2}$, $x_i=x$ $(i=1,2,\ldots,m)$ in Eq. (3.3a) gives Eq. (3.3b) which is a generalization of the known result for $m=1$ [8, vol. 1, p. 743, Eq. (4)]. Eq. (3.4a) is obtained by setting $\mu=-\frac{1}{2}$ in Eq. (3.2) in which case the first term in braces on the right of the latter equation vanishes. Note that by setting $y=\frac{1}{2}\pi$ in Eq. (3.4a), we deduce Eq. (2.4) from which Eqs. (2.5)–(2.9) may be obtained by further specialization. In addition, by letting $v_i=\frac{1}{2}$, $x_i=x$ $(i=1,2,\ldots,m)$ in Eq. (3.4a), we obtain Eq. (3.4b) which is a generalization of Eq. (2.6) and of the result for $m=1$, $0<x<y<\pi$ in [8, vol. 1, p. 743, Eq. (3)], where the misprint $-\frac{1}{2}$ should be replaced by $-\frac{1}{2}x$. Eq. (3.5b) may be obtained from Eq. (3.5a) (or vice versa) by interchanging x with y and μ with v. Either result is recorded in [8, vol. 2, p. 683, Eq. (7)], but the conditions for its validity appear to be incorrect. In Eq. (3.5a) if we let $x\to y$ (or in Eq. (3.5b) let $y\to x$) and use Gauss's summation theorem for ${}_2F_1(1)$, we obtain Eq. (3.5c) which is given (now with correct conditions for its validity) in [8, vol. 2, p. 683, Eq. (9)]. The limiting process just used must be justified, however, since we have only assumed that the sum on the left hand side of Eq. (3.5a), considered as a function of x and y, is actually continuous at $x=y$. We shall do this now by considering the latter infinite sum in detail.

Thus, from Eqs. (1.7) and (3.1) we have

$$\sum_{n=1}^{\infty}\frac{J_\mu(2ny)J_v(2ny)}{n^{\mu+v}}=-\frac{y^{\mu+v}}{2\Gamma(\mu+1)\Gamma(v+1)}+\int_0^\infty\frac{J_\mu(2yt)J_v(2yt)}{t^{\mu+v}}\,\mathrm{d}t+2\sum_{k=1}^{\infty}\int_0^\infty\frac{\cos(2\pi kt)J_\mu(2yt)J_v(2yt)}{t^{\mu+v}}\,\mathrm{d}t, \tag{3.6}$$

where the latter integral converges provided that $\mathrm{Re}(\mu+v)>-1$. However, the former integral of Weber and Schafheitlin converges for $\mathrm{Re}(\mu+v)>0$ and is equal to the first term on the right hand side of Eq. (3.5c) (see [13, p. 403, Eq. (2)]). Therefore, Eq. (3.6) is valid for $\mathrm{Re}(\mu+v)>0$, in which case, the sum is absolutely convergent. The analysis and evaluation of the third term on the right hand side of Eq. (3.6) is similar to that of the summation in Eq. (2.10). Thus, omitting the details we obtain for $\mathrm{Re}(\mu+v)>0$, $y>0$:

$$\begin{aligned}\sum_{n=1}^{\infty}&\frac{J_\mu(2ny)J_v(2ny)}{n^{\mu+v}}\\ &=-\frac{y^{\mu+v}}{2\Gamma(\mu+1)\Gamma(v+1)}+\frac{\sqrt{\pi}\Gamma(\mu+v)y^{\mu+v-1}}{2\Gamma(\mu+\frac{1}{2})\Gamma(v+\frac{1}{2})\Gamma(\mu+v+\frac{1}{2})}\\ &\quad+\frac{\sqrt{\pi}\Gamma(\mu+v)y^{\mu+v-1}}{\Gamma(\mu+\frac{1}{2})\Gamma(v+\frac{1}{2})\Gamma(\mu+v+\frac{1}{2})}\sum_{k=1}^{p}{}_3F_2\left[\begin{matrix}\frac{1}{2}-\mu-v,\frac{1}{2}-\mu,\frac{1}{2}-v;\ \frac{k^2\pi^2}{4y^2}\\ \frac{1-\mu-v}{2},\frac{2-\mu-v}{2};\end{matrix}\right]\\ &\quad+\frac{\pi^{\mu+v+\frac{1}{2}}}{2y}\frac{\Gamma(-\frac{\mu+v}{2})}{\Gamma(\frac{1-\mu+v}{2})\Gamma(\frac{1+\mu-v}{2})\Gamma(\frac{1+\mu+v}{2})}\\ &\quad\times\sum_{k=1}^{p}k^{\mu+v}{}_4F_3\left[\begin{matrix}\frac{1-\mu-v}{2},\frac{1+\mu-v}{2},\frac{1-\mu+v}{2},\frac{1+\mu+v}{2};\ \frac{k^2\pi^2}{4y^2}\\ \frac{1}{2},\frac{\mu+v}{2},\frac{2+\mu+v}{2};\end{matrix}\right]\end{aligned}$$

$$+\frac{2^{\mu+\nu}\pi^{\mu+\nu+1}(\mu+\nu)}{y^2}\,\frac{\sin(\frac{\pi}{2}(\mu+\nu))\Gamma(-1-\mu-\nu)}{\Gamma(\frac{\mu-\nu}{2})\Gamma(\frac{\nu-\mu}{2})}$$

$$\times\sum_{k=1}^{p} k^{\mu+\nu+1}{}_3F_2\left[\begin{matrix}\frac{2-\mu-\nu}{2},\frac{2+\mu-\nu}{2},\frac{2-\mu+\nu}{2};\ \frac{k^2\pi^2}{4y^2}\\ \frac{3}{2},\frac{3+\mu+\nu}{2};\end{matrix}\right], \tag{3.7}$$

where p is a nonnegative integer such that $\frac{1}{2}\pi p < y < \frac{1}{2}\pi(p+1)$. When $0 < y < \frac{1}{2}\pi$, then $p = 0$, there are no contributions from the three k-summations, and Eq. (3.7) reduces to Eq. (3.5c). Glasser has obtained a result [3, Eq. (13)] which reduces to Eq. (3.5c).

Finally, we may set $m = 0$ in Eq. (3.2) provided that we define $F_C^{(0)} \equiv 1$. Thus, we obtain

$$\sum_{n=1}^{\infty}\frac{J_\mu(2ny)}{n^\mu} = -\frac{y^\mu}{2\Gamma(\mu+1)} + \frac{\sqrt{\pi}y^{\mu-1}}{2\Gamma(\mu+\frac{1}{2})},$$

where $\operatorname{Re}\mu > -\frac{1}{2}$, $0 < y < \pi$. This is a specialization of the result [12, p. 65, example (vi)]

$$\sum_{n=1}^{\infty}\frac{J_\mu(2ny)}{n^\mu} = -\frac{y^\mu}{2\Gamma(\mu+1)} + \frac{\sqrt{\pi}y^{\mu-1}}{2\Gamma(\mu+\frac{1}{2})} + \frac{\sqrt{\pi}y^{-\mu}}{\Gamma(\mu+\frac{1}{2})}\sum_{k=1}^{p}(y^2-\pi^2k^2)^{\mu-1/2},$$

where $\operatorname{Re}\mu > -\frac{1}{2}$, $y > 0$ and p is a nonnegative integer such that $\pi p < y < \pi(p+1)$. By noting Eq. (1.3), the latter may be written in hypergeometric form as

$$\sum_{n=1}^{\infty}{}_0F_1[-;\mu+1;-n^2y^2] = -\frac{1}{2} + \frac{\sqrt{\pi}}{2y}\frac{\Gamma(\mu+1)}{\Gamma(\mu+\frac{1}{2})} + \frac{\sqrt{\pi}}{y^{2\mu}}\frac{\Gamma(\mu+1)}{\Gamma(\mu+\frac{1}{2})}\sum_{k=1}^{p}(y^2-\pi^2k^2)^{\mu-1/2} \tag{3.8}$$

which, along with Eq. (1.4), we shall generalize in the next section.

4. Sums containing ${}_1F_2[-n^2x^2]$

We define for $x > 0$,

$$S(x) \equiv \sum_{n=1}^{\infty}(-1)^n{}_1F_2[\alpha;\beta,\mu+1;-n^2x^2], \tag{4.1}$$

$$T(x) \equiv \sum_{n=1}^{\infty}{}_1F_2[\alpha;\beta,\mu+1;-n^2x^2]. \tag{4.2}$$

Since for $|z| \to \infty$, $|\arg z| < \frac{1}{2}\pi$

$${}_1F_2[\alpha;\beta,\gamma;-z^2] \sim \frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta-\alpha)\Gamma(\gamma-\alpha)}z^{-2\alpha} + \frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\frac{1}{2})\Gamma(\alpha)}z^{1/2+\alpha-\beta-\gamma}\cos\left[2z+\frac{\pi}{2}\left(\frac{1}{2}+\alpha-\beta-\gamma\right)\right]$$

(see, e.g., [6, Eq. (2.8)]), it is not difficult to see that the sums in Eqs. (4.1) and (4.2) converge absolutely provided that $\frac{1}{2} < \operatorname{Re}\alpha < \operatorname{Re}(\beta+\mu-\frac{1}{2})$.

Now by applying Eq. (1.5) and the method of proof of Eq. (2.1) mutatis mutandis to $S(x)$ we obtain

$$S(x) = -\frac{1}{2} + 2\sum_{k=1}^{\infty} \int_0^{\infty} \cos(\pi(2k-1)t)\,{}_1F_2[\alpha;\beta,\mu+1;-x^2t^2]\,\mathrm{d}t. \tag{4.3}$$

In order to proceed further, we shall need the cosine transform of the hypergeometric function ${}_1F_2[-b^2t^2]$, which has been derived in an elementary way in [7]:

$$\begin{aligned}\int_0^{\infty} &\cos(2at)\,{}_1F_2[\alpha;\beta,\gamma;-b^2t^2]\,\mathrm{d}t\\
&= \frac{\sqrt{\pi}}{2b}\frac{\Gamma(\alpha-\frac{1}{2})\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta-\frac{1}{2})\Gamma(\gamma-\frac{1}{2})}\,{}_2F_1\left[\begin{matrix}\frac{3}{2}-\beta,\frac{3}{2}-\gamma; & \dfrac{a^2}{b^2}\\ \frac{3}{2}-\alpha; & \end{matrix}\right]\\
&\quad+\frac{\sqrt{\pi}}{2a}\left(\frac{a^2}{b^2}\right)^{\alpha}\frac{\Gamma(\frac{1}{2}-\alpha)\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta-\alpha)\Gamma(\gamma-\alpha)}\,{}_2F_1\left[\begin{matrix}1+\alpha-\beta,1+\alpha-\gamma; & \dfrac{a^2}{b^2}\\ \frac{1}{2}+\alpha; & \end{matrix}\right],\end{aligned} \tag{4.4}$$

where $0<\operatorname{Re}\alpha<\operatorname{Re}(\beta+\gamma-\frac{1}{2})$ and $0<a<b$; when $0<b<a$, the integral vanishes. If $a=0$, $b>0$, the result holds for $\frac{1}{2}<\operatorname{Re}\alpha<\operatorname{Re}(\beta+\gamma-\frac{1}{2})$, the second term on the right-hand side vanishes, and the Gaussian function preceding it reduces to unity.

Thus, by applying the latter result to the integral in Eq. (4.3) we deduce for $x>0$

$$\begin{aligned}\sum_{n=1}^{\infty}&(-1)^n\,{}_1F_2[\alpha;\beta,\mu+1;-n^2x^2]\\
&=-\frac{1}{2}+\frac{\sqrt{\pi}}{x}\frac{\Gamma(\beta)\Gamma(\alpha-\frac{1}{2})\Gamma(\mu+1)}{\Gamma(\alpha)\Gamma(\beta-\frac{1}{2})\Gamma(\mu+\frac{1}{2})}\sum_{k=1}^{p}{}_2F_1\left[\begin{matrix}\frac{3}{2}-\beta,\frac{1}{2}-\mu; & \dfrac{(k-\frac{1}{2})^2\pi^2}{x^2}\\ \frac{3}{2}-\alpha; & \end{matrix}\right]\\
&\quad+\frac{1}{\sqrt{\pi}}\frac{\Gamma(\beta)\Gamma(\frac{1}{2}-\alpha)\Gamma(\mu+1)}{\Gamma(\alpha)\Gamma(\beta-\alpha)\Gamma(\mu-\alpha+1)}\left(\frac{\pi^2}{x^2}\right)^{\alpha}\\
&\quad\times\sum_{k=1}^{p}\left(k-\frac{1}{2}\right)^{2\alpha-1}{}_2F_1\left[\begin{matrix}1+\alpha-\beta,\alpha-\mu; & \dfrac{(k-\frac{1}{2})^2\pi^2}{x^2}\\ \frac{1}{2}+\alpha; & \end{matrix}\right],\end{aligned} \tag{4.5}$$

where $0<\operatorname{Re}\alpha<\operatorname{Re}(\beta+\mu+\frac{1}{2})$ and p is a nonnegative integer such that $(p-\frac{1}{2})\pi<x<(p+\frac{1}{2})\pi$. When $\alpha=\beta$, it is easy to see that the third term on the right hand side in Eq. (4.5) vanishes so that this result reduces immediately to Eq. (1.4).

Next, by applying Eq. (1.5) to the sum in Eq. (4.2) we get

$$T(x) = -\frac{1}{2} + \int_0^{\infty} {}_1F_2[\alpha;\beta,\mu+1;-x^2t^2]\,\mathrm{d}t + 2\sum_{k=1}^{\infty}\int_0^{\infty}\cos(2\pi kt)\,{}_1F_2[\alpha;\beta,\mu+1;-x^2t^2]\,\mathrm{d}t.$$

Again the integrals are evaluated by using Eq. (4.4), where for convergence of the latter integral $0<\operatorname{Re}\alpha<\operatorname{Re}(\beta+\mu+\frac{1}{2})$. The former integral converges provided that $\frac{1}{2}<\operatorname{Re}\alpha<\operatorname{Re}(\beta+\mu+\frac{1}{2})$. Thus for $x>0$, we deduce

$$\begin{aligned}
&\sum_{n=1}^{\infty} {}_1F_2[\alpha;\beta,\mu+1;-n^2x^2] \\
&\quad = -\frac{1}{2}+\frac{\sqrt{\pi}}{2x}\frac{\Gamma(\beta)\Gamma(\alpha-\frac{1}{2})\Gamma(\mu+1)}{\Gamma(\alpha)\Gamma(\beta-\frac{1}{2})\Gamma(\mu+\frac{1}{2})} \\
&\qquad +\frac{\sqrt{\pi}}{x}\frac{\Gamma(\beta)\Gamma(\alpha-\frac{1}{2})\Gamma(\mu+1)}{\Gamma(\alpha)\Gamma(\beta-\frac{1}{2})\Gamma(\mu+\frac{1}{2})}\sum_{k=1}^{p} {}_2F_1\left[\begin{matrix}\frac{3}{2}-\beta,\frac{1}{2}-\mu; \\ \frac{3}{2}-\alpha;\end{matrix}\ \frac{k^2\pi^2}{x^2}\right] \\
&\qquad +\frac{1}{\sqrt{\pi}}\frac{\Gamma(\beta)\Gamma(\frac{1}{2}-\alpha)\Gamma(\mu+1)}{\Gamma(\alpha)\Gamma(\beta-\alpha)\Gamma(\mu-\alpha+1)}\left(\frac{\pi^2}{x^2}\right)^{\alpha} \\
&\qquad \times\sum_{k=1}^{p} k^{2\alpha-1} {}_2F_1\left[\begin{matrix}1+\alpha-\beta,\alpha-\mu; \\ \frac{1}{2}+\alpha;\end{matrix}\ \frac{k^2\pi^2}{x^2}\right],
\end{aligned} \tag{4.6}$$

where $\frac{1}{2}<\operatorname{Re}\alpha<\operatorname{Re}(\beta+\mu+\frac{1}{2})$ and p is a nonnegative integer such that $\pi p<x<\pi(p+1)$. When $\alpha=\beta$, the above fourth term on the right-hand side vanishes so that Eq. (4.6) reduces immediately to Eq. (3.8).

Since Lommel functions $s_{\mu,\nu}(z)$ and Struve functions $\boldsymbol{H}_\mu(z)$ are related to each other and to ${}_1F_2[-z^2/4]$ via

$$s_{\mu,\nu}(z)=\frac{z^{\mu+1}}{(\mu+\nu+1)(\mu-\nu+1)}\,{}_1F_2\left[1;\frac{\mu-\nu+3}{2},\frac{\mu+\nu+3}{2};\frac{-z^2}{4}\right]$$

and

$$\boldsymbol{H}_\mu(z)=\frac{s_{\mu,\mu}(z)}{\sqrt{\pi}\,2^{\mu-1}\Gamma(\mu+\frac{1}{2})},$$

from Eqs. (4.5) and (4.6) upon specialization we arrive at the following for $x>0$, $\operatorname{Re}\mu>-\frac{3}{2}$:

$$\begin{aligned}
&\sum_{n=1}^{\infty}(-1)^n\frac{s_{\mu,\nu}(2nx)}{n^{\mu+1}} \\
&\quad = -\frac{2^{\mu}x^{\mu+1}}{(\mu+\nu+1)(\mu-\nu+1)} \\
&\qquad +2^{\mu-1}\pi x^{\mu}\frac{\Gamma(\frac{\mu+\nu+1}{2})\Gamma(\frac{\mu-\nu+1}{2})}{\Gamma(\frac{\mu+\nu+2}{2})\Gamma(\frac{\mu-\nu+2}{2})}\sum_{k=1}^{p} {}_2F_1\left[\begin{matrix}\frac{-\mu-\nu}{2},\frac{-\mu+\nu}{2}; \\ \frac{1}{2};\end{matrix}\ \frac{(k-\frac{1}{2})^2\pi^2}{x^2}\right] \\
&\qquad -2^{\mu}\pi^2x^{\mu-1}\sum_{k=1}^{p}\left(k-\frac{1}{2}\right){}_2F_1\left[\begin{matrix}\frac{1-\mu-\nu}{2},\frac{1-\mu+\nu}{2}; \\ \frac{3}{2};\end{matrix}\ \frac{(k-\frac{1}{2})^2\pi^2}{x^2}\right],
\end{aligned} \tag{4.7}$$

where p is a nonnegative integer such that $(p-\frac{1}{2})\pi<x<(p+\frac{1}{2})\pi$;

$$\sum_{n=1}^{\infty}\frac{s_{\mu,\nu}(2nx)}{n^{\mu+1}}=-\frac{2^{\mu}x^{\mu+1}}{(\mu+\nu+1)(\mu-\nu+1)}+2^{\mu-2}\pi x^{\mu}\frac{\Gamma(\frac{\mu+\nu+1}{2})\Gamma(\frac{\mu-\nu+1}{2})}{\Gamma(\frac{\mu+\nu+2}{2})\Gamma(\frac{\mu-\nu+2}{2})}$$
$$+2^{\mu-1}\pi x^{\mu}\frac{\Gamma(\frac{\mu+\nu+1}{2})\Gamma(\frac{\mu-\nu+1}{2})}{\Gamma(\frac{\mu+\nu+2}{2})\Gamma(\frac{\mu-\nu+2}{2})}\sum_{k=1}^{p}{}_2F_1\left[\begin{matrix}\frac{-\mu-\nu}{2},\frac{-\mu+\nu}{2}; & \dfrac{k^2\pi^2}{x^2}\\ \frac{1}{2}; & \end{matrix}\right]$$
$$-2^{\mu}\pi^2x^{\mu-1}\sum_{k=1}^{p}k\,{}_2F_1\left[\begin{matrix}\frac{1-\mu-\nu}{2},\frac{1-\mu+\nu}{2}; & \dfrac{k^2\pi^2}{x^2}\\ \frac{3}{2}; & \end{matrix}\right], \tag{4.8}$$

where p is a nonnegative integer such that $\pi p<x<\pi(p+1)$;

$$\sum_{n=1}^{\infty}(-1)^n\frac{\boldsymbol{H}_{\mu}(2nx)}{n^{\mu+1}}=-\frac{x^{\mu+1}}{\sqrt{\pi}\Gamma(\mu+\frac{3}{2})}+\frac{\pi x^{\mu}p}{\Gamma(\mu+1)}$$
$$-\frac{2\pi^{3/2}x^{\mu-1}}{\Gamma(\mu+\frac{1}{2})}\sum_{k=1}^{p}\left(k-\frac{1}{2}\right){}_2F_1\left[\begin{matrix}\frac{1}{2}-\mu,\frac{1}{2}; & \dfrac{(k-\frac{1}{2})^2\pi^2}{x^2}\\ \frac{3}{2}; & \end{matrix}\right], \tag{4.9}$$

where p is a nonnegative integer such that $(p-\frac{1}{2})\pi<x<(p+\frac{1}{2})\pi$;

$$\sum_{n=1}^{\infty}\frac{\boldsymbol{H}_{\mu}(2nx)}{n^{\mu+1}}=-\frac{x^{\mu+1}}{\sqrt{\pi}\Gamma(\mu+\frac{3}{2})}+\frac{\pi x^{\mu}}{\Gamma(\mu+1)}\left(p+\frac{1}{2}\right)$$
$$-\frac{2\pi^{3/2}x^{\mu-1}}{\Gamma(\mu+\frac{1}{2})}\sum_{k=1}^{p}k\,{}_2F_1\left[\begin{matrix}\frac{1}{2}-\mu,\frac{1}{2}; & \dfrac{k^2\pi^2}{x^2}\\ \frac{3}{2}; & \end{matrix}\right], \tag{4.10}$$

where p is a nonnegative integer such that $\pi p<x<\pi(p+1)$.

When x is such that $p=0$, Eqs. (4.5) and (4.6) give, respectively,

$$\sum_{n=1}^{\infty}(-1)^n{}_1F_2[\alpha;\beta,\mu+1;-n^2x^2]=-\frac{1}{2},\quad 0<\operatorname{Re}\alpha<\operatorname{Re}\left(\beta+\mu+\frac{1}{2}\right),\quad |x|<\frac{1}{2}\pi;$$

$$\sum_{n=1}^{\infty}{}_1F_2[\alpha;\beta,\mu+1;-n^2x^2]=-\frac{1}{2}+\frac{\sqrt{\pi}}{2x}\frac{\Gamma(\beta)\Gamma(\alpha-\frac{1}{2})\Gamma(\mu+1)}{\Gamma(\alpha)\Gamma(\beta-\frac{1}{2})\Gamma(\mu+\frac{1}{2})},$$
$$\frac{1}{2}<\operatorname{Re}\alpha<\operatorname{Re}\left(\beta+\mu+\frac{1}{2}\right),\quad 0<x<\pi.$$

For $\operatorname{Re}\mu>-\frac{3}{2}$, Eqs. (4.7)–(4.10) give, respectively, when $p=0$,

$$\sum_{n=1}^{\infty}(-1)^n\frac{s_{\mu,\nu}(2nx)}{n^{\mu+1}}=-\frac{2^{\mu}x^{\mu+1}}{(\mu+\nu+1)(\mu-\nu+1)},\quad |x|<\frac{1}{2}\pi;$$

$$\sum_{n=1}^{\infty}\frac{s_{\mu,\nu}(2nx)}{n^{\mu+1}}=-\frac{2^{\mu}x^{\mu+1}}{(\mu+\nu+1)(\mu-\nu+1)}$$

$$+2^{\mu-2}\pi x^{\mu}\frac{\Gamma(\frac{\mu+\nu+1}{2})\Gamma(\frac{\mu-\nu+1}{2})}{\Gamma(\frac{\mu+\nu+2}{2})\Gamma(\frac{\mu-\nu+2}{2})}, \quad 0<x<\pi;$$

$$\sum_{n=1}^{\infty}(-1)^n\frac{\boldsymbol{H}_{\mu}(2nx)}{n^{\mu+1}} = -\frac{x^{\mu+1}}{\sqrt{\pi}\Gamma(\mu+\frac{3}{2})}, \quad |x|<\frac{1}{2}\pi;$$

$$\sum_{n=1}^{\infty}\frac{\boldsymbol{H}_{\mu}(2nx)}{n^{\mu+1}} = -\frac{x^{\mu+1}}{\sqrt{\pi}\Gamma(\mu+\frac{3}{2})} + \frac{\pi x^{\mu}}{2\Gamma(\mu+1)}, \quad 0<x<\pi.$$

The latter two results are recorded in [8, vol. 3, p. 399, Eq. (3)].

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